A Gentle Introduction to the Boundary Element Method in Matlab/Freemat

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Abstract: - The *Boundary Element Method* is developed in its most simple form; for the solution of *Laplace's equatio* in an interior domain with a straight line approximation to the boundary. The direct and indirect approaches to th *boundary element method* are included. The methods are developed in *Freemat,* a language similar to *Matlab.*

The codes for the solution of *Laplace's equation* in a general domain with a general (Robin) boundary condition a developed. The codes are applied to a typical test problem. The codes are made available as open source and can b downloaded from www.east-lancashire-research.org.uk (report AR-08-14).

Keywords—boundary element method, BEM, Laplace's equation.

1 Introduction

The boundary element method (BEM) is an important computational analysis technique that engineers and scientists can apply to a range of problems. The purpose of this article and the accompanying software is to meet the needs of scientists and engineers who are somewhat unfamiliar with the BEM, but have an understanding of numerical methods and computer programming, or would like to apply the BEM to appropriate engineering problems with minimal fuss.

The application of the boundary element method to an appropriate scientific or engineering problem essentially requires a mesh of the boundary of the domain only, and the determination of the boundary condition on the surface. The computational solution then yields the approximate solution at selected points in the domain. The BEM is generally more efficient to apply and execute than competing methods, such as the finite element or finite difference methods. Hence the application of the BEM presents an attractive option to scientists and engineers. The authors are developing an MSc in Engineering Computation (at ELIHE www.elihe.ac.uk) and this package is expected to act as a teaching aid on one of the modules in that course. A simple notation is used to assist in the understanding of the development of the BEM.

In this work, the so-called *direct* and *indirect* BEMs for the solution of the interior Laplace equation are developed. This is the most straightforward problem to which the BEM can be applied. Laplace's equation also models a number of physical phenomena, such as steady state heat conduction and electrostatics. There is substantial recent research on the application of the BEM to Laplace's equation [1-8].

Over recent decades, Matlab [9] has become an increasingly important language for scientific computation. Freemat [10] is a freely available alternative compiler for Matlab. All codes are developed in Freemat, but they can be also used in the Matlab environment. Matlab/Freemat is based on Matrix arithmetic, allowing an economy of coding and naturally allows parallel processing, if it is available. All software is open source and can be downloaded via the websites [10-12].

2 The Interior Laplace Equation

The Laplace equation is the simplest elliptic PDE. It is one of the equations of potential theory and they have been received extensive mathematical analysis. It also serves as model elliptic equations for learning, implementing and testing numerical methods. In this article we are using the Laplace equation in order to motivate our understanding of the properties and practice of the BEM. In this paper the BEM is developed to solve the two-dimensional Laplace Equation

$$\frac{\partial^2\varphi(\mathbf{p})}{\partial x^2}+\frac{\partial^2\varphi(\mathbf{p})}{\partial y^2}=0 \quad (\mathbf{p}\in D)$$

or in the shorthand form:

$$\nabla^2\varphi(\mathbf{p})=0 \qquad (\mathbf{p}\in D) \quad (1)$$

in an interior domain D, with an enclosing boundary S, as illustrated in figure 1. A boundary condition is determined on S. For this work we assume that the boundary condition is of the *Robin* or mixed form:

$$a(\mathbf{p})\varphi(\mathbf{p}) + b(\mathbf{p})\frac{\partial\varphi}{\partial n_p}(\mathbf{p}) = f(\mathbf{p}) \quad (\mathbf{p} \in S), \quad (2)$$

where $a(\mathbf{p})$, $b(\mathbf{p})$ and $f(\mathbf{p})$ are real-valued functions defined on *S* only and n_p is the unit outward normal to the boundary at p (assumed to be unique). The general boundary condition includes the Dirichlet (essential) boundary condition ($a(\mathbf{p}) = 1, b(\mathbf{p}) = 0$) and Neumann (derivative) boundary condition ($a(\mathbf{p}) = 0, b(\mathbf{p}) = 1$).

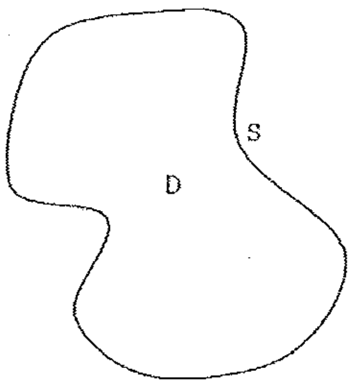

Fig 1. Illustration of the domain.

Together, the governing PDE within a domain (eg (1)) and the boundary condition (2) is called a boundary value problem (BVP). The solution to such a problem is principally the determination of φ (at points) in the domain *D*, whether by analytic or numerical methods.

3 Integral Equation Reformulation

The first stage in the development of a BEM from a boundary value problem (like (1)-(2)) is to rewrite the partial differential equation as an integral equation. Traditionally, there have been two ways of doing this; the *direct* method and the *indirect* method. In this section we will go through the stages for developing the integral equation formulations

3.1Green's function

In order to do this it is useful to introduce an *influence function*; a function that determines the effect at a point q of a unit source at a point **p**, this function is also often known as a *Green's function*.

For the two-dimensional Laplace equation (1), the Green's function is known to be

$$G(\mathbf{p},\mathbf{q}) = \frac{-1}{2\pi}\ln(|\mathbf{p}-\mathbf{q}|). \quad (3)$$

The Green's function has the property $\nabla^2 G(\mathbf{p}-\mathbf{q}) = \delta(\mathbf{p}-\mathbf{q})$ where δ is the Dirac delta function.

3.2 Laplace Integral Operators

As a further set of building blocks, it is also useful to define the set of *Laplace integral operators*:

$$\{L\zeta\}_\Gamma(\mathbf{p}) = \int_\Gamma G(\mathbf{p},\mathbf{q})\zeta(\mathbf{q})dS_q, \quad (4a)$$

$$\{M\zeta\}_\Gamma(\mathbf{p}) = \int_\Gamma \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\zeta(\mathbf{q})dS_q, \quad (4b)$$

$$\{M'\zeta\}_\Gamma(\mathbf{p};\mathbf{w}) = \frac{\partial}{\partial w}\int_\Gamma G(\mathbf{p},\mathbf{q})\zeta(\mathbf{q})dS_q = \int_\Gamma \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial w}\zeta(\mathbf{q})dS_q \quad (4c)$$

$$\{N\zeta\}_\Gamma(\mathbf{p};\mathbf{w}) = \frac{\partial}{\partial w}\int_\Gamma \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\zeta(\mathbf{q})dS_q \quad (4d)$$

where Γ is the whole or any part of *S*, ζ is any real-valued function, defined on Γ.

When **p**εΓ, then we have the more particular form of *M'* and *N*:

$$\{M'\zeta\}_\Gamma(\mathbf{p};\mathbf{n}_p) = \frac{\partial}{\partial n_p}\int_\Gamma G(\mathbf{p},\mathbf{q})\zeta(\mathbf{q})dS_q = \int_\Gamma \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_p}\zeta(\mathbf{q})dS_q \quad (4e)$$

$$\{N\zeta\}_\Gamma(\mathbf{p};\mathbf{n}_p) = \frac{\partial}{\partial n_p}\int_\Gamma \frac{\partial G(\mathbf{p},\mathbf{q})}{\partial n_q}\zeta(\mathbf{q})dS_q \quad (4f)$$

Note that the derivative $\frac{\partial}{\partial n_p}$ cannot always be taken directly inside the integral in (4f), if we did then the integral can be *hypersingular*, and therefore not defined in the normal sense, when **q**=**p**. *N* is therefore not a true integral operator, but belongs to the more general class of *pseudo-differential* operators. For convenience, we continue to refer to *L,M,M'* and *N* as integral operators, but we will also keep the special case on *N* in mind.

Applying *L* and *N* to any function ζ, any boundary Γ and any vector $\mathbf{n}_p$ gives rise to a continuous function in space. However the operators *M* and *M'* have jump discontinuities at the boundary.

$$\lim_{\varepsilon\to 0}\{M\zeta\}_\Gamma(\mathbf{p}-\varepsilon\mathbf{n}_p) - \frac{1}{2}\zeta(\mathbf{p}) = \{M\zeta\}_\Gamma(\mathbf{p}) \quad (5a)$$

$$\lim_{\varepsilon\to 0}\{M'\zeta\}_\Gamma(\mathbf{p}-\varepsilon\mathbf{n}_p;\mathbf{n}_p) + \frac{1}{2}\zeta(\mathbf{p}) = \{M'\zeta\}_\Gamma(\mathbf{p};\mathbf{n}_p) \quad (5b)$$

where **p**εΓ and Γ is smooth at **p**.

ISSN: 1790-2769

ISBN: 978-960-474-012-3

3.3 Direct Method

The following equation arises as a result of Green's second theorem

$$\{M\varphi\}_S(\mathbf{p}) - \{Lv\}_S(\mathbf{p}) = -\varphi(\mathbf{p}) \quad (\mathbf{p}\varepsilon D). \quad (6a)$$

where $v = \dfrac{\partial\varphi(\mathbf{p})}{\partial n_p}$. For points on S we apply the limit (5a) in equation (6a):

$$\{M\varphi\}_S(\mathbf{p}) - \{Lv\}_S(\mathbf{p}) = -\frac{1}{2}\varphi(\mathbf{p}) \quad (\mathbf{p}\varepsilon S). \quad (6b)$$

Given these equations, the method of solution would involve solving (6b) with the boundary condition (2) in order to find approximations to φ and v on the boundary and then use equation (6a) to compute φ at any chosen points in the domain.

However, there is one notable case when this method will not work as well. In the case of a pure Dirichlet boundary condition, (6a) is effectively a Fredholm integral equation of the first kind. It is well known that the numerical solution of first kind equations is not as efficient as it is for the equivalent second kind equation (which equation (6b) would otherwise be) [14].

We can easily introduce another equation, using the Laplace integral operators, that will be useful to us. Differentiating (6a) with respect to a vector **w**, gives:

$$\{N\varphi\}_S(\mathbf{p};\mathbf{w}) - \{M'v\}_S(\mathbf{p};\mathbf{w}) = -\frac{\partial\varphi}{\partial w}(\mathbf{p}) \quad (\mathbf{p}\varepsilon D). \quad (7a)$$

For points **p** near the boundary, with $\mathbf{n}_p$ being the unique unit outward normal at **p**, then (7a) becomes

$$\{N\varphi\}_S(\mathbf{p};\mathbf{n}_p) - \{M'v\}_S(\mathbf{p};\mathbf{n}_p) = -\frac{\partial\varphi}{\partial n_p}(\mathbf{p}) \quad (\mathbf{p}\varepsilon D) \quad (7b)$$

Moving the point **p** to the surface and applying the limit (5b) gives rise to the following equation on the surface:

$$\{N\varphi\}_S(\mathbf{p};\mathbf{n}_p) - \{M'v\}_S(\mathbf{p};\mathbf{n}_p) = -\frac{1}{2}v(\mathbf{p}) \quad (8)$$

One disadvantage in using equation (8) as a basis for solving the Laplace equation, is that it now contains the *hypersingular* operator N. The other disadvantage is that if we wish to solve the Neumann problem using equation (8) then we have to solve over the operator N, which leads to a similar loss of efficiency that is found in solving first kind equations.

In order to avoid the problems with the Dirichlet problem with equation (6b) and the Neumann problem with equation (8), a hybrid equation is proposed

$$\{(M + \frac{1}{2}I + \mu N)\}\varphi(\mathbf{p}) = \{(L + (M' - \frac{1}{2}I))\}v(\mathbf{p}) \quad (9)$$

For suitable weighting parameter μ, equation (9) forms a suitable basis for solving the Robin BVP and the special cases of the Dirichlet and Neumann BVPs.

Once approximations to φ and v are found on the boundary from equation (8) with the boundary condition (2), we can use equation (6a) to determine and approximation to φ for any point (**p**) in the domain.

3.4 Indirect Method

The alternative or *indirect* approach to obtaining an integral reformulation of the PDE involves writing the solution φ as a layer potential. The most obvious way of doing this is to write

$$\varphi(\mathbf{p}) = \{L\sigma\}_S(\mathbf{p}) \quad (\mathbf{p}\varepsilon D \cup S) \quad (10a)$$

where σ is a density function defined on S

It is possible to solve the Dirichlet problem from equation (10a). This would normally involve finding σ on S by solving the integral equation (10a). However, the same equation cannot be used for the Neumann problem and, what is more, it requires solution to be carried out over the first kind operator L.

By differentiating the equation (10a) with respect to any vector **w**, we obtain

$$\frac{\partial\varphi}{\partial w}(\mathbf{p}) = \frac{\partial}{\partial w}\{L\sigma\}(\mathbf{p}) = \{M'\sigma\}(\mathbf{p};\mathbf{w}) \quad (\mathbf{p}\varepsilon D) \quad (10b)$$

As **p** approaches the boundary and we take $\mathbf{w}=\mathbf{n}_p$. and on the boundary equation (10b) becomes

$$\frac{\partial\varphi}{\partial n_p}(\mathbf{p}) = v(\mathbf{p}) = \frac{\partial}{\partial n_p}\{L\sigma\}(\mathbf{p}) = \{M'\sigma\}(\mathbf{p};\mathbf{n}_p) + \frac{1}{2}\sigma(\mathbf{p}) \quad (\mathbf{p}\varepsilon S) \quad (10c)$$

where the jump discontinuity (5b) has been included.

Equation (10c) relates v on S to σ. Hence equation (10c) can be used as a basis for solving the Neumann problem. It is a second kind equation and so it is very suitable as a basis for solution.

We do not have a more general solution method. To do this let us introduce a hybrid single- and double-layer potential:

$$\varphi = \{L\sigma_\mu\}(\mathbf{p}) + \mu\{M\sigma_\mu\}(\mathbf{p}) \quad (\mathbf{p}\varepsilon D) \quad (11a)$$

where σ_μ is a density function that depends on the choice of μ. By allowing the point **p** approach the boundary S, equation (11a) becomes:

$$\varphi = \{L\sigma_\mu\}(\mathbf{p}) + \mu\{M\sigma_\mu\}(\mathbf{p}) - \frac{\mu}{2}\sigma_\mu(\mathbf{p}) \quad (\mathbf{p}\varepsilon S) \quad (11b)$$

For μ≠0 equation (11b) is a suitable equation to solve the Dirichlet problem since it is always a second-kind integral equation. However, for the Neumann and more general Robin problem it is useful to introduce another equation that is the outcome of differentiating equation (11a), firstly with respect to any vector **w**:

$$\frac{\partial\varphi}{\partial w}(\mathbf{p}) = \frac{\partial}{\partial w}\{L\sigma_\mu\}(\mathbf{p}) + \mu\frac{\partial}{\partial w}\{M\sigma_\mu\}(\mathbf{p}) \; (\mathbf{p}\varepsilon D). \qquad (11c)$$

Allowing **p** to approach the boundary and **w** becomes the unit outward normal to the boundary there gives the following equations:

$$\frac{\partial\varphi}{\partial n_p}(\mathbf{p}) = \frac{\partial}{\partial n_p}\{L\sigma_\mu\}(\mathbf{p}) + \mu\frac{\partial}{\partial n_p}\{M\sigma_\mu\}(\mathbf{p}) + \frac{\mu}{2}\sigma_\mu(\mathbf{p}) \; . (\mathbf{p}\varepsilon S) \qquad (11d)$$

Using the operator notation, this give

$$v(\mathbf{p}) = \{M'\sigma_\mu\}(\mathbf{p}) + \mu\{N\sigma_\mu\}(\mathbf{p}) - \frac{\mu}{2}\sigma_\mu(\mathbf{p}) \; . (\mathbf{p}\varepsilon S) \qquad (11e)$$

Substituting the expressions for φ and v in equation (11a) and (11b) into the general Robin boundary condition (2) gives the following BIE:

$$\{\alpha(L + \mu(M - \frac{1}{2}I)) + \beta((M' + \frac{1}{2}I) + \mu N)\}\sigma_\mu(\mathbf{p}) = f(\mathbf{p}) \qquad (11f)$$

Equation (11e) is most suitable for the solution of the classes of boundary conditions considered.

The indirect BEM involves solving (11e) to return an approximation to σ_μ on the boundary. Equation (11a) can then be employed to compute an approximation to φ in the domain *D*.

4 The Discrete Operators

There is a variety of techniques for deriving the system of linear equations from a given integral equation [15]. Collocation is one of the most straightforward and popular and it is the one that we will be using.

4.1 Collocation

The application of collocation to a BIE requires that the boundary is represented by a set of *panels*. For example a two dimensional boundary can be approximated by a set of straight lines, as illustrated in figure 2.

In order to complete the discretisation of the integral equations, the boundary functions also need to be approximated on each panel. It is the characteristics of the panel and the representation of the boundary function on the panel that together define the *element* in the BEM. By representing the boundary functions by a characteristic form on each panel, the BIEs can be simplified into a linear system of equations. Most simply, the boundary functions can be approximated by a constant on each panel. The collocation (or representative) point is at the centre of the panel (C^1collocation). The overall process is that of discretising the integral operators and the methods for carrying this out are covered in reference [16].

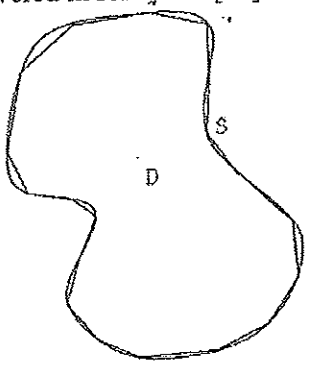

Fig 2. Illustration of the boundary divided into panels.

Let the ΔSj (for j = 1, 2, .., n) be the n panels that represent an approximation to S in the method. We may write $S \approx \tilde{S} = \sum_{j=1}^{n} \Delta S_j$. (12)

Following from equation (4a), we may write

$$\{L\zeta\}_S(\mathbf{p}) = \int_S G(\mathbf{p},\mathbf{q})\zeta(\mathbf{q})dS_q \approx \int_{\tilde{S}} G(\mathbf{p},\mathbf{q})\zeta(\mathbf{q})dS_q$$

$$= \sum_{j=1}^{n} \int_{\Delta\tilde{S}} G(\mathbf{p},\mathbf{q})\zeta(\mathbf{q})dS_q \approx \sum_{j=1}^{n} \zeta_j \int_{\Delta\tilde{S}_j} G(\mathbf{p},\mathbf{q})dS_q$$

$$= \sum_{j=1}^{n} \zeta_j \{Le\}_{\Delta S_j}(\mathbf{p})$$

where in the final expression we have made the approximation $\zeta(\mathbf{q}) \approx \zeta_j$ (a constant) on the j[th] panel and *e* is the unit function. A similar discretisation can be applied to the other integral operators

$$\{M\zeta\}_S(\mathbf{p}) \approx \sum_{j=1}^{n} \zeta_j \{Me\}_{\Delta S_j}(\mathbf{p}) \qquad (13b)$$

$$\{M'\zeta\}_S(\mathbf{p}) \approx \sum_{j=1}^{n} \zeta_j \{M'e\}_{\Delta S_j}(\mathbf{p}), \quad (13c)$$

and $\{N\zeta\}_S(\mathbf{p}) \approx \sum_{j=1}^{n} \zeta_j \{Ne\}_{\Delta S_j}(\mathbf{p})$. (13d)

For any point **p**, $\{Le\}_{\Delta S_j}(\mathbf{p})$, $\{Me\}_{\Delta S_j}(\mathbf{p})$, $\{M'e\}_{\Delta S_j}(\mathbf{p})$ and $\{Ne\}_{\Delta S_j}(\mathbf{p})$ are termed the discrete Laplace integral operators.

4.2 Simplifying the integrands

Writing G(**p**,**q**) as G(r) where r=|**r**| and **r**=**q**-**p**, the Green's function (3) can be written as follows:

$$G(r) = \frac{-1}{2\pi}\ln(r). \quad (14)$$

The derivatives of G with respect to r:

$$\frac{\partial}{\partial r}G(r) = \frac{-1}{2\pi}\frac{1}{r} \quad (15) \text{ and}$$

$$\frac{\partial^2}{\partial r^2}G(r) = \frac{1}{2\pi}\frac{1}{r^2}. \quad (16)$$

The normal derivatives of G:

$$\frac{\partial G}{\partial n_q} = \frac{\partial G}{\partial r}\frac{\partial r}{\partial n_q}, \quad (17)$$

$$\frac{\partial G}{\partial n_p} = \frac{\partial G}{\partial r}\frac{\partial r}{\partial n_p}, \quad (18)$$

and $$\frac{\partial^2 G}{\partial n_p \partial n_q} = \left(\frac{\partial G}{\partial r}\frac{\partial^2 r}{\partial n_p \partial n_q} + \frac{\partial^2 G}{\partial r^2}\frac{\partial r}{\partial n_p}\frac{\partial r}{\partial n_q}\right). \quad (19)$$

The normal derivatives of r:

$$\frac{\partial r}{\partial n_q} = -\frac{\mathbf{r}.\mathbf{n}_q}{r}, \quad (20)$$

$$\frac{\partial r}{\partial n_p} = \frac{\mathbf{r}.\mathbf{n}_p}{r}, \quad (21)$$

and $$\frac{\partial^2 r}{\partial n_p \partial n_q} = -\frac{1}{r}\left(\mathbf{n}_p.\mathbf{n}_q + \frac{\partial r}{\partial n_p}\frac{\partial r}{\partial n_q}\right). \quad (22)$$

4.3 Evaluating the integrals

In most cases of evaluating the discrete integrals in (13a-d), the integrand is continuous and can be approximated most efficiently by Gaussian quadrature. For the case in which **p** lies on the element of integration, the integral can be evaluated by a simple formulae. Let the element Δ have length a+b with the point **p** lying a distance a from one end and a distance b from the other, as illustrated in the figure 3.

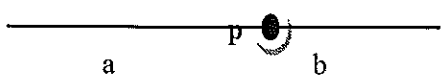

Figure 3. A general element with the collocation point **p** lying on the element.

In the case illustrated in figure 3 we can derive the following expressions:

$$\{Le\}_\Delta(\mathbf{p}) = \frac{1}{2\pi}(a + b - a\log a - b\log b), \quad (23a)$$

$$\{Me\}_\Delta(\mathbf{p}) = 0, \ (23b) \qquad \{M'e\}_\Delta(\mathbf{p}) = 0, \ (23c)$$

$$\{Ne\}_\Delta(\mathbf{p}) = \frac{1}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right). \quad (23d)$$

5 Matrices in the BEM

The BEM is derived by applying an integral equation method to the appropriate BIE. The most straightforward method to apply is that of collocation.

The initial development of the application of collocation to give expressions for the discrete Laplace operators is given in Section 5. To continue the development, approximations to the (unknown) boundary functions – whether that be φ and/or v in the direct method, or σ in the indirect method – are obtained by replacing the BIE by a matrix-vector equation and then solving it. Approximations to φ in the domain can then be found by direct integration.

In the collocation method, the general point **p** in the BIE (e.g. equations (9) for the direct method or equation (11e) for the indirect method), takes the value of every central point on each panel; the collocation points: $\mathbf{p}=\mathbf{p}_{S1}, \mathbf{p}_{S2},\ldots, \mathbf{p}_{Sn}$. For illustration, let us apply the collocation method to BIE (10a) (for **p**εS):

$$\varphi(\mathbf{p}) = \{L\sigma\}_S(\mathbf{p}) \ (\mathbf{p}\varepsilon S) \quad (24)$$

Following the development in equation (13a), the following discrete form of equation (24) is obtained:

$$\varphi(\mathbf{p}) = \{L\sigma\}_S(\mathbf{p}) \approx \sum_{j=1}^{n}\sigma_j\{Le\}_{\Delta S_j}(\mathbf{p}).$$

Allowing **p** to take the value of $\mathbf{p}_{Si}$, a collocation points gives the following:

$$\varphi(\mathbf{p}_{Si}) = \{L\sigma\}_S(\mathbf{p}_{Si}) \approx \sum_{j=1}^{n}\sigma_j\{Le\}_{\Delta S_j}(\mathbf{p}_{Si}).$$

Let us now introduce the notation:

$$\varphi_{Si} = \varphi(\mathbf{p}_{Si}), v_{Si} = v(\mathbf{p}_{Si}), \sigma_{Si} = \sigma(\mathbf{p}_{Si}),$$

$$\underline{\varphi}_S = \begin{bmatrix}\varphi_{S1}\\ \varphi_{S2}\\ \vdots\\ \vdots\\ \varphi_{Sn}\end{bmatrix}, \quad \underline{v}_S = \begin{bmatrix}v_{S1}\\ v_{S2}\\ \vdots\\ \vdots\\ v_{Sn}\end{bmatrix}, \quad \underline{\sigma}_S = \begin{bmatrix}\sigma_{S1}\\ \sigma_{S2}\\ \vdots\\ \vdots\\ \sigma_{Sn}\end{bmatrix},$$

$$[\mathbf{L}_{SS}]_{ij} = \{Le\}_{\Delta S_j}(\mathbf{p}_{Si}), \ [\mathbf{M}_{SS}]_{ij} = \{Me\}_{\Delta S_j}(\mathbf{p}_{Si}),$$

$$[\mathbf{M}^t{}_{SS}]_{ij} = \{M'e\}_{\Delta S_j}(\mathbf{p}_{Si}), [\mathbf{N}_{SS}]_{ij} = \{Ne\}_{\Delta S_j}(\mathbf{p}_{Si}).$$

Returning to the integral equation (24): by applying the collocation method it is then replaced by the following equation: $\underline{\varphi}_S \approx \mathbf{L}_{SS}\underline{\sigma}_S$. Which can be solved for the Dirichlet case to return an approximation to σ on S. (Although basing the BEM on this first kind equation is not advised; this is meant to be illustrative.)

Usually, the objective is to find the solution φ in the domain. Returning to the BIE (10a) (for pεD):

$$\varphi(\mathbf{p}) = \{L\sigma\}_S(\mathbf{p})\ (\mathbf{p}\varepsilon D) \quad (25)$$

For points **p** in the domain, we can approximate φ(**p**) as before $\varphi(\mathbf{p}) = \{L\sigma\}_S(\mathbf{p}) \approx \sum_{j=1}^{n} \sigma_j \{Le\}_{\Delta S_j}(\mathbf{p})$ except in its application, an approximation to σ (the values of the σ_i) has been obtained.

Let the solution be sought at the *m* domain points **p** = $\mathbf{p}_{D1}, \mathbf{p}_{D2}, \ldots, \mathbf{p}_D$

$$\varphi(\mathbf{p}_{Di}) = \{L\sigma\}_S(\mathbf{p}_{Di}) \approx \sum_{j=1}^{n} \sigma_j \{Le\}_{\Delta S_j}(\mathbf{p}_{Di}).$$

Let us now introduce the notation:

$$\varphi_{Di} = \varphi(\mathbf{p}_{Di}), v_{Di} = v(\mathbf{p}_{Di}), \sigma_{Di} = \sigma(\mathbf{p}_{Di}),$$

$$\underline{\varphi}_D = \begin{bmatrix} \varphi_{D1} \\ \varphi_{D2} \\ \vdots \\ \vdots \\ \varphi_{Dm} \end{bmatrix}, \ \underline{v}_D = \begin{bmatrix} v_{D1} \\ v_{D2} \\ \vdots \\ \vdots \\ v_{Dn} \end{bmatrix},$$

$$[\mathbf{L}_{DS}]_{ij} = \{Le\}_{\Delta S_j}(\mathbf{p}_{Di}),\ [\mathbf{M}_{DS}]_{ij} = \{Me\}_{\Delta S_j}(\mathbf{p}_{Di}),$$

$$[\mathbf{M}^t{}_{DS}]_{ij} = \{M^t e\}_{\Delta S_j}(\mathbf{p}_{Di}), [\mathbf{N}_{DS}]_{ij} = \{Ne\}_{\Delta S_j}(\mathbf{p}_{Di}).$$

Returning back to the example integral equation (24), using this notation, approximations to the solution at the domain points can be determined by the following matrix-vector multiplication: $\underline{\varphi}_D \approx \mathbf{L}_{DS}\underline{\sigma}_S$. It can be observed that there is a general case in evaluating the required matrices; it is given a set of points and a description of the boundary, a method for evaluating the matrix components is required.

6 Boundary Element Method

Having developed all the building blocks we may now complete the coding for the BEMs. In section IV, two classes of BIE were introduced and these lead to two classes of BEM; the direct and the indirect method.

6.1 Direct Method

For the chosen integral equation (9), the discrete analogue for collocation points on S is as follows:

$$[\mathbf{M}_{SS}+\tfrac{1}{2}\mathbf{I}+\mu\mathbf{N}_{SS}]\ \underline{\varphi}_S = [\mathbf{L}_{SS}+\mu(\mathbf{M}^t{}_{SS}-\tfrac{1}{2}\mathbf{I})]\ \underline{v}_S, \quad (25)$$

with the general Robin boundary condition

$$\alpha_i\varphi_i + \beta_i v_i = f_i \text{ for } i=1..n. \quad (26)$$

Once the solution on the boundary is found to (25), (26), the solution at any domain point can be found through integrating over the boundary using (6a).

6.2 Indirect Method

For the chosen integral equation (11e), the discrete analogue for collocation points on S is as follows:

$$\{(\mathbf{L}_{SS}+\mu(\mathbf{M}_{SS}-\tfrac{1}{2}\mathbf{I}))\mathbf{D}_\alpha+((\mathbf{M}^t{}_{SS}+\tfrac{1}{2}\mathbf{I})+\mu\mathbf{N}_{SS})\mathbf{D}_\beta\}\underline{\sigma}_\mu=\underline{f} \quad (27)$$

where the general Robin boundary condition is included such that **Dα** and $\mathbf{D}_\beta$ are diagonal matrices with diagonal components α_i and β_i respectively. The solution of equation (27) gives $\underline{\sigma}_\mu$, the discrete equivalent of the layer potential σ_μ on the boundary. The solution in the domain and on the boundary can then be found by direct integration using equations (11a) and (11b).

6.3 Weighting parameter

The value of the weighting parameter μ is arbitrary from the mathematical point of view. However, from the computational point of view, we need to avoid the equation becoming close to a first kind equation (that is if μ is small for the Dirichlet boundary condition) and we also want to avoid similar issues arising if we solve over something close to the *N* operator (that is if μ is large for the Neumann boundary condition).

The "size" or norm of the relevant matrices must also be taken into account when choosing a value for μ. A reasonable choice would therefore seem to be to choose a value of μ that balances the relevant matrices and therefore the relative contribution from the two underlying formulations. Out of the four matrices in the boundary solution, only $\mathbf{N}_{SS}$ has the property such that its norm is inversely proportional to the size of the panels; the norms of the other matrices stay approximately the same as the boundary panels become smaller. Hence, in the direct and indirect methods, the underlying contributions from the two integral equations are balanced through applying the following values for μ: $\mu = \dfrac{\left\|\mathbf{M}'_{SS} + \frac{1}{2}\mathbf{I}\right\|}{\left\|\mathbf{N}_{SS}\right\|}$ for the direct method

and $\mu = \dfrac{\left\|\mathbf{M}_{SS} + \frac{1}{2}\mathbf{I}\right\|}{\left\|\mathbf{N}_{SS}\right\|}$ for the indirect method.

7 Test Problem and Results

Finally the direct and indirect BEMs are applied a test problem and the results are observed. The test problem consists of a square with vertices (0,0), (0.1,0), (0.1, 0.1) and (0.1,0), as illustrated in figure 4. The boundary condition is defined as illustrated in figure 5. A computational solution is sought at the points (0.025,0.025), (0.025,0.075), (0.075,0.075), (0.075,0.025) and (0.05,0.05), also

ISSN: 1790-2769

ISBN: 978-960-474-012-3

illustrated in figure 4.

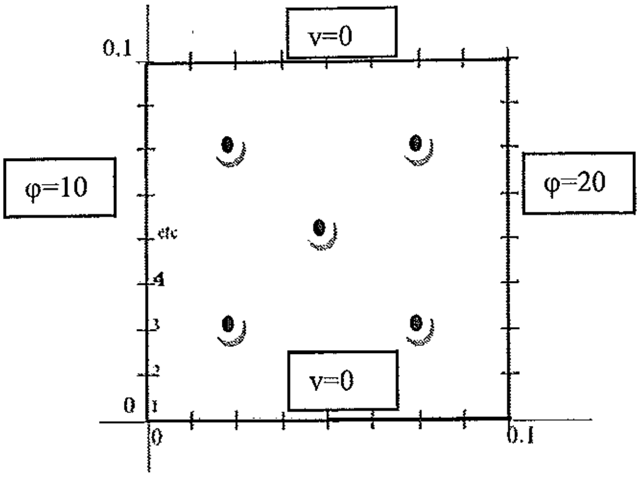

Figure 4. A square divided into 32 panels.

The results from the two methods are given in the following table.

point	*exact*	*direct*	*indirect*
(0.025,0.025)	12.5	12.4709	12.4891
(0.025,0.075)	12.5	12.4709	12.4891
(0.05,0.05)	15	15.0008	14.9939
(0.075,0.025)	17.5	17.5306	17.4927
(0.075,0.075)	17.5	17.5306	17.4927

References

[1] R. Cunderlik , K. Mikula, AND M. Mojzes, A comparison of the variational solution to the Neumann geodetic boundary value problem with the geopotential model EGM-96. Contributions to Geophysics and Geodesy **34**(3), 2004, Available: http://www.math.sk/mikula/cmm.pdf

[2] Guror R, Olsen B, Kroese A, Cook F, Kumar S (2004). Evaluation of critical components of non-ceramic insulators in-service: role of defective interfaces, Power Systems Engineering Research Center, Arizona State University. Available:http://www.pserc.org/cgi-pserc/getbig/publicatio/reports/2004report/psercreport.pdf

[3] P. Lazic, H. Stefancic, H. Abraham,The Robin Hood method: a novel numerical method for electrostatic problems based on a non-local charge transfer , *Journal of Comp. Phys.* **213**(1) , pp117 - 140, 2006 Available:http://arxiv.org/PS_cache/physics/pdf/0411/0411192v1.pdf

[4] B. Lim, K, L. Venkatachalam, A, Jahangir, S. B. Johnson, S. J. Asirvatham, Concurrent Application of Charge Using a Novel Circuit Prevents Heat-Related Coagulum Formation During Radiofrequency Ablation, *Journal of Cardiovascular Electrophysiology* , pp1-8. doi:10.1111/j.1540-8167.2008.01117.x

[5] F. Seydou, T. Seppanen, O. Ramahi, Numerical solution of 3D Laplace and Helmholtz equations for parallel scatterers, *Antennas and Prop. Soc. Int. Symp.,* IEEE. **2B**(3-8) July, pp97 - 100, 2005.

[6] A. VanderWyst, A. Christlieb, M. Sussman, I. D. Boyd, Simulation of Charge and Mass Distributions of Indium Droplets Created by Field Emission, *AIAA*. Available: http://hpcc.engin.umich.edu/CFD/research/NGPD/Publications/AIAA-06-3560.pdf

[7] S. M. Kirkup, DC capacitor simulation by the boundary element method, *Comm. in Num. Meth. in Eng.* **23**(9), 855 - 869, 2007. Available [17].

[8] S. M. Kirkup and D. J. Henwood, An empirical error analysis of the boundary element method applied to Laplace's equation, *App. Math. Mod.*, **18**(1) , pp 32-38, 1994. Available [17].

[9] Freemat http://freemat.sourceforge.net

[10] Matlab http://www.mathworks.com

[11] http://www.kirkup.info/opensource

[12] http://www.east-lancashire-research.org.uk Report AR-08-14.

[13] http://www.boundary-element-method.com

[14] L. M. Delves and J. L. Mohamed, *Computational Methods for Integral Equations*, Cambridge University Press, 1985.

[15] C. A. Brebbia. *The Boundary Element Method for Engineers*. John Wiley and Sons, 1978.

[16] S. M. Kirkup. Fortran Codes for Computing the Discrete Helmholtz Integral Operators, Advances in Computational Mathematics, **9**, pp391-409, 1998.Available [17].

[17] http://www.kirkup.info/papers